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THE MIXTURE WEIBULL-GENERALIZED GAMMA DISTRIBUTION

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Abstract

The mixture distribution is defined as one of the most important ways to obtain new probability distributions in applied probability and several research areas. It is a compounding of statistical distributions, which arises when sampling is from inhomogeneous populations (or mixed populations) with a different probability density function in each component. Finite mixture models also play a vital role in life testing and reliability. According to the previous reasons, we have been looking for more flexible alternative to the lifetime data. This paper introduces a new mixed distribution, namely the Mixture Weibull-Generalized Gamma distribution, which is obtained by

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mixing Weibull and generalized gamma distributions. We refer to the new distribution as (W-GG) distribution. The new model contains twenty eight lifetime distributions as special cases such as the Weibull, Lindley, Quasi Lindley, Janardan, Gamma, Rayleigh and exponential distributions, among others. The properties of the W-GG distribution are discussed and the maximum likelihood estimation is used to evaluate the parameters. Explicit expressions are derived for the moments and examine the order statistics. This model is capable of modeling various shapes of aging and failure criteria.

1. Introduction

There are many distributions for modeling the lifetime data among the known parametric models; the most popular are the gamma, lognormal and the Weibull distributions.

For more than half a century the Weibull distribution introduced by Weibull [28] has attracted the attention of statisticians working on theory and methods as well as in various fields of applied statistics. It is of utmost interest to theory orientated statisticians because of its great number of special features and to practitioners because of its ability to fit to data from various fields, ranging from life data to weather data or observations made in economics and business administration, in hydrology, in biology or in the engineering sciences.

This distribution does not provide a good fit to data sets with bathtub shaped or upside-down bathtub shaped (unimodal) failure rates, often encountered in reliability, engineering and biological studies. Hence a number of new distributions modeling the data in a better way have been constructed in literature as modifications of this distribution. Several distributions have been proposed in the literature to extend the Weibull distribution (see Pham and Lai [20]) which presents a review of some of the generalizations or modifications of Weibull distribution.

While the generalized gamma (GG) distribution introduced by Stacy [26] includes the exponential, Weibull, gamma and Rayleigh distributions, among others as its special models, is the most popular model for analyzing skewed data and presents a flexible family in the varieties of shapes and hazard functions for modeling duration.

The inferential procedures for the GG distribution are difficult perhaps because of an additional shape parameter. However, the statistical analysis of its parameters based on complete as well as censored samples have been studied by many authors for details, see (Maswadah et al. [17]) and its references.

According to the great features and limitation on the Weibull and generalized gamma distributions, we introduced a new mixed distribution, namely the Mixture Weibull-Generalized Gamma (W-GG) distribution, which is obtained by mixing Weibull and generalized gamma distributions hoping to reduce this limitation and introduced a more fixable model hoping to decrease the great hole between the statistical models in fitting the different types of data.

Weibull [28] introduced the Weibull distribution with cumulative distribution function (cdf) denoted by $G_1(x)$ and probability density function (pdf) denoted by $g_1(x)$ (for $x > 0$) as

$$G_1(x) = 1 - e^{-(\alpha x)^{\beta}}, \tag{1}$$

and

$$g_1(x) = \alpha\beta(\alpha x)^{\beta-1} e^{-(\alpha x)^{\beta}}, \tag{2}$$

respectively, where $\beta > 0$ is shape parameter and $\alpha > 0$ is scale parameter.

Stacy [26] introduced the generalized gamma distribution with cdf denoted by $G_2(x)$ and pdf denoted by $g_2(x)$ (for $x > 0$) as

$$G_2(x) = \frac{\gamma(\lambda, (\alpha x)^{\beta})}{\Gamma(\lambda)} = P(\lambda, (\alpha x)^{\beta}), \tag{3}$$

and

$$g_2(x) = \frac{\alpha\beta}{\Gamma(\lambda)} (\alpha x)^{\lambda\beta-1} e^{-(\alpha x)^{\beta}}, \tag{4}$$

respectively, where $\beta > 0$ and $\lambda > 0$ are shape parameters, $\alpha > 0$ is scale parameter and $\gamma(a, x) = \int_0^x u^{a-1} e^{-u} du$ represents the lower incomplete gamma function.

Based on a finite number of pdfs denoted by $g_i(x)$ one can introduce a new pdf denoted by $f(x)$ using the mixture distribution with weights a_i as:

$$f(x) = \sum_{i=1}^{n} a_i \times g_i(x), \tag{5}$$

where

$$\sum_{i=1}^{n} a_i = 1.$$

Finite mixture models also play a vital role in life testing and reliability. Size-frequency distributions in animal populations with distinct age-groups, the distribution of times to failure in a mixture of good and defective items, the distribution of some diagnostic measure in a mixed population of patients and some of whom have a given disease and some of whom do not, are all examples of mixed distributions. Böhning et al. [6] introduced a very good review of the mixture models.

Based on the pdfs of the Weibull and generalized gamma distributions given by (2) and (4), and on the mixture distribution given by (5) with weight $a_1 = \frac{\theta\alpha}{\theta\alpha + 1}$ and $a_2 = \frac{1}{\theta\alpha + 1}$, then the pdf of the W-GG model is given by

$$\begin{aligned} f(x) &= a_1 g_1(x) + a_1 g_1(x) \\ &= \frac{\alpha\beta}{\theta\alpha + 1} (\alpha x)^{\beta - 1} e^{-(\alpha x)^{\beta}} \left[\theta\alpha + \frac{(\alpha x)^{\beta(\lambda - 1)}}{\Gamma(\lambda)} \right], \end{aligned} \tag{6}$$

whereas its cdf can be expressed as

$$F(x) = \frac{1}{\theta\alpha + 1} [\theta\alpha(1 - e^{-(\alpha x)^{\beta}}) + P(\lambda, (\alpha x)^{\beta})]. \tag{7}$$

The failure rate function associated to (7) is given by

$$h(x) = \frac{\alpha\beta(\alpha x)^{\beta-1} e^{-(\alpha x)^{\beta}} \left[\theta\alpha + \frac{1}{\Gamma(\lambda)}(\alpha x)^{\beta(\lambda-1)}\right]}{1 + \theta\alpha e^{-(\alpha x)^{\beta} - P(\lambda,\, (\alpha x)^{\beta})}}. \tag{8}$$

Another importance of the proposed W-GG model is that it is very flexible model that approaches to different distributions when its parameters are changed. The flexibility of the W-GG is explained in Table 1 where it has 28 sub-models when their parameters are carefully chosen.

Table 1. The special cases of the W-GG distribution.

Distribution	Parameters				Author
	α	β	λ	θ	
GG				0	Stacy [26]
2P-G		1		0	
Erlang		1	Integer	0	
χ^2	$1/2$	1	$n/2$	0	
Scaled χ^2	$1/a\sqrt{2}$	1	$n/2$		
Maxwell	$1/a\sqrt{2}$	2	1	0	See Bekker and Roux [5]
Nakagami	$\sqrt{m/w}$	2	m	0	See Shankar et al. [21]
Generalized half normal	$\frac{1}{\theta 2^{1/2\sigma}}$	$\frac{1}{2\sigma}$	$\frac{1}{2}$	0	Cooray and Ananda [8]
Standard normal (folded)	$\frac{1}{\sqrt{2}}$	2	$\frac{1}{2}$	0	
W			1	0	Weibull [28]
R		2	1	0	
E		1	1	0	
L		1	2	1	

QL		1	2	η/α	Shanker Mishra [22]
2P-L		2	2	$1/\eta$	Shanker et al. [23]
J	a/b	1	2	$1/b$	Shanker et al. [24]
New Mixture W-GG				1	
Mixture W-Erlang			Integer		
Mixture W-W			1		
Mixture W- χ^2	$1/2$	1	$n/2$		
Mixture W-Scaled χ^2	$1/a\sqrt{2}$	1	$n/2$		
Mixture Maxwell-Maxwell	$\frac{1}{a\sqrt{2}}$	2	1		
Mixture R-Nakagami	$\sqrt{m/w}$	2	m	0	
Mixture R-half normal	$\frac{1}{\sqrt{2}}$	2	$\frac{1}{2}$		
Mixture R-R		2	1		
Mixture E-2P-G		1			
Mixture E-Erlang		1	Integer		
Mixture E-E		1	1		

Note: GG, Generalized gamma; G, Gamma; J. Janardan; 2P, Two parameter; 1P, One parameter; M, Maxwell; W, Weibull; χ^2, chi-square; R, Rayleigh; E, exponential, L; Lindley; Q, Quasi.

Figures 1 (a), (b), (c) and (d) provide some plots of the W-GG density curves for selected values of θ, α, β and λ. The figures shows a monotonic increasing and decreasing, uni-modal and bimodal shapes. The pdf is turns from a uni-modal to bi-modal shapes at different values of the shape parameter λ for given values of the other remaining parameters.

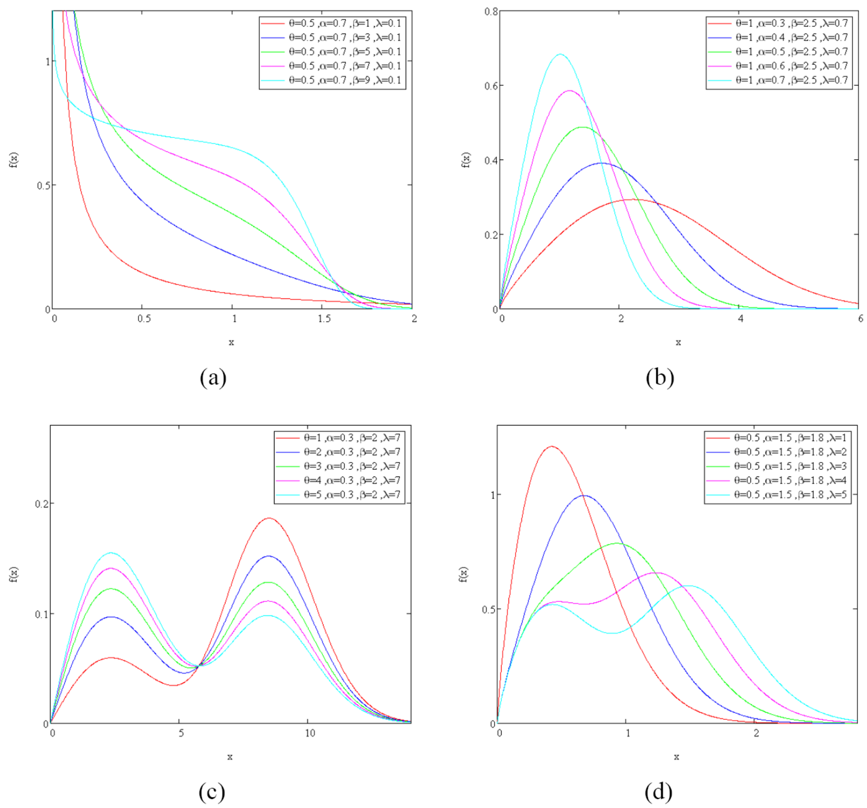


Figure 1. Plots of the W-GG density function for some parameter values. (a) For different values of β with $\theta = 0.5, \alpha = 0.7$ and $\lambda = 0.1$. (b) For different values of α with $\theta = 1, \beta = 2.5$ and $\lambda = 0.7$. (c) For different values of θ with $\alpha = 0.3, \beta = 2$ and $\lambda = 7$. (d) For different values of λ with $\theta = 0.5, \alpha = 0.7$ and $\beta = 0.1$.

Figure 2 does the same for the associated hazard rate function, showing that it is quite flexible for modelling survival data.

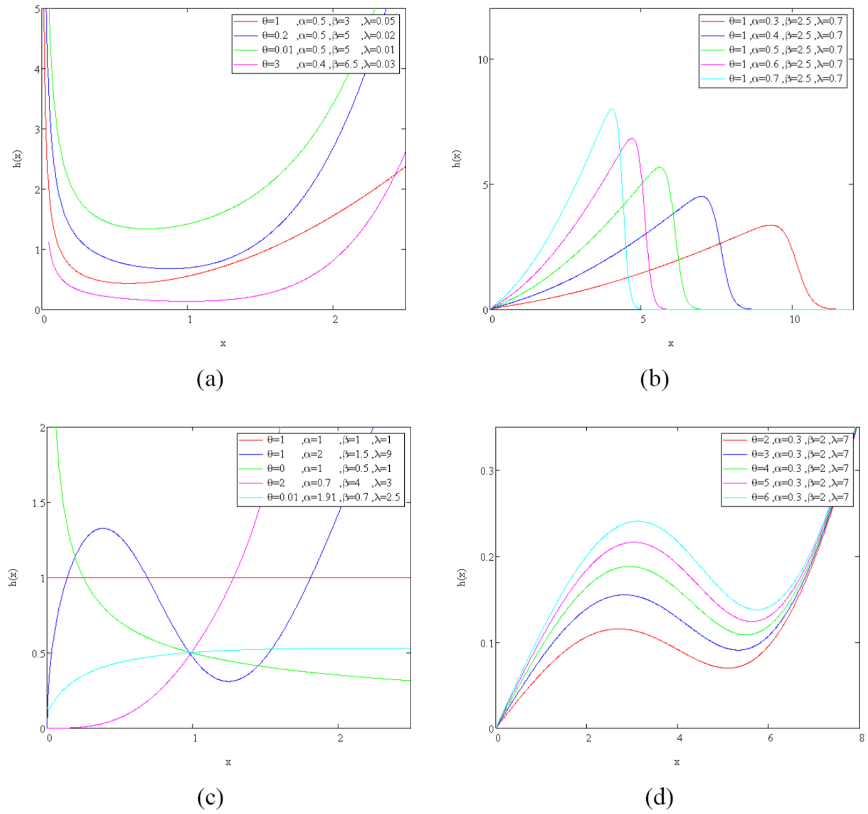


Figure 2. (a), (b), (c) and (d) are plots of the hazard rate function for some parameter values.

The rest of the article is organized as follows. In Section 2, we derive an expansion to the pdf. In Section 3, we obtain the rth non-central moments. Section 4 gives the quantile function for the new model. In Section 5, we introduce the order statistics and its moments. While, the probability weighted moments are given in Section 6. Mean deviation and its application Bonferroni and Lorenz curves are given in Section 7. While Section 8, gives entropies measures such as Rényi and Shannon. Residual life and reversed residual life functions are given in Sections 9 and 10. Section 11 introduces the method of likelihood estimation as point estimation and the confidence interval as an interval estimation of the unknown parameters then, gives the equation used to estimate the parameters using the maximum product spacing estimates and the least square estimates techniques. Finally, we fit the distribution to real data set to examine it.

2. Expansion for the pdf Function

In this section we introduce another expression for the pdf function using the Maclaurin.

From (6) and using the expansion

$$e^{x} = \sum_{i=0}^{\infty} \frac{x^{i}}{i!}, \tag{9}$$

and applying it to the term $e^{-(\alpha x)^{\beta}}$, then the pdf of the W-GG after some simplifications can be written as:

$$f(x) = \frac{\alpha\beta}{\theta\alpha + 1} \sum_{i=0}^{\infty} \frac{(-1)^{i}}{i!} \left[\theta\alpha(\alpha x)^{i\beta+\beta-1} + \frac{1}{\Gamma(\lambda)} (\alpha x)^{i\beta+\beta\lambda-1} \right]. \tag{10}$$

Another motivation to the new model based on (10), for positive integers β and λ, the new model can represent a summation of infinite of two polynomials.

3. Moments

The rth non-central moments, $E(x^{r}) = \mu^{r} = \int_{0}^{\infty} x^{r} f(x) dx$, or (moments about the origin) of the W-GG using (6) is given by:

$$\mu^{r} = \frac{\alpha^{-k}}{\theta\alpha + 1} \left[\theta\alpha \cdot \Gamma\left(\frac{k}{\beta} + 1 \right) + \frac{1}{\Gamma(\lambda)} \Gamma\left(\lambda + \frac{k}{\beta} \right) \right]. \tag{11}$$

4. Quantile Function

The quantile function is obtained by inverting the cumulative distribution (7), and then the pth quantile x_{p} of the W-GG model is the real solution of the following equation:

$$[(\theta\alpha + 1)p - \theta\alpha]\Gamma(\lambda) = \gamma(\lambda, (\alpha x)^{\beta}) - \theta\alpha\Gamma(\lambda)e^{-(\alpha x)^{\beta}}. \tag{12}$$

An expansion for the median M follows by taking $p = 0.5$.

5. Moment of Order Statistics

Order statistics make their appearance in many areas of statistical theory and practice. Let the random variable $X_{r:n}$ be the rth order statistic $(X_{1:n}, X_{2:n}, ..., X_{n:n})$ in a sample of size n from the W-GG distribution. The pdf of $X_{r:n}$ for $r = 1, 2, ..., n,$ is given by

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x)[F_X(x)]^{r-1}[1 - F_X(x)]^{n-r}, \; x > 0, \tag{13}$$

where $f(\cdot)$ and $F(\cdot)$ represent the pdf and the cdf of the W-GG distribution, respectively. Substituting from (6) and (7) into (13) gives

$$f_{r:n}(x) = \frac{\alpha\beta(\alpha x)^{\beta-1} n!}{(r-1)!(n-r)!} e^{-(\alpha x)^{\beta}} \left[\theta\alpha + \frac{(\alpha x)^{\beta(\lambda-1)}}{\Gamma(\lambda)}\right]$$

$$\times [1 + \theta\alpha e^{-(\alpha x)^{\beta}} - P(\lambda, (\alpha x)^{\beta})]^{n-r}$$

$$\times \frac{1}{(\theta\alpha + 1)^n} \left[\theta\alpha(1 - e^{-(\alpha x)^{\beta}}) + \frac{P(\lambda, (\alpha x)^{\beta})}{\Gamma(\lambda)}\right]^{r-1}.$$

Also the cdf of $X_{r:n}$, $F_{r:n}(x) = \sum_{k=r}^{n} \binom{n}{k} [F_X(x)]^k [1 - F_X(x)]^{n-k}$, is given by

$$F_{r:n}(x) = \frac{1}{(\theta\alpha + 1)^n} \sum_{k=r}^{n} \binom{n}{k} [\theta\alpha(1 - e^{-(\alpha x)^{\beta}}) + P(\lambda, (\alpha x)^{\beta})]^k$$

$$\times [1 + \theta\alpha e^{-(\alpha x)^{\beta}} - P(\lambda, (\alpha x)^{\beta})]^{n-k}.$$

The kth moment about zero of the rth order statistic is obtained by using a result due to Barakat and Abdelkader [4] and becomes

$$E(X_{r:n}^{k}) = k\sum_{i=n-r+1}^{\infty}(-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i}\int_{0}^{\infty}x^{k-1}[1-F(x)]^{i}\,dt,$$

where

$$[1-F(x)]^{i} = \frac{1}{(\theta\alpha+1)^{i}}[1+\theta\alpha e^{-(\alpha x)^{\beta}} - P(\lambda, (\alpha x)^{\beta})]^{i}.$$

Applying the binomial expansion twice to the term $[1+\theta\alpha e^{-(\alpha x)^{\beta}} - P(\lambda, (\alpha x)^{\beta})]^{i}$, then $[1-F(x)]^{i}$ after some simplifications can be written as:

$$[1-F(x)]^{i}$$

$$= \frac{1}{(\theta\alpha+1)^{i}}\sum_{j=0}^{i}\sum_{l=0}^{j}\binom{i}{j}\binom{j}{l}\frac{(-1)^{l}(\theta\alpha)^{j-l}}{[\Gamma(\lambda)]^{l}}e^{-(j-l)(\alpha x)^{\beta}}[\gamma(\lambda, (\alpha x)^{\beta})]^{h}. \quad (14)$$

for a power series raised to a positive integer n (see Gradshteyn and Ryzhik [14]) gives an expansion as

$$\left(\sum_{i=0}^{\infty}a_i u^{i}\right)^{n} = \sum_{i=0}^{\infty}c_{n,i}u^{i}, \quad (15)$$

where the coefficients $c_{n,i}$ (for $i = 1, 2, \cdots$) are easily determined from the recurrence equation $C_{n,i} = (ia_0)^{-1}\sum_{m=1}^{i}[m(n+1)-i]a_m c_{n,i-m}$ and $C_{n,0} = a_0^{n}$.

Applying (15) on $[\gamma(\lambda, (\alpha x)^{\beta})]^{k}$ by using the expansion $\gamma(s, x) = \sum_{h=0}^{\infty}(-1)^{h}\frac{x^{s+h}}{h!(s+h)}$, then $[1-F(x)]^{i}$ can be written as

$$[1-F(x)]^{i}$$

$$= \frac{1}{(\theta\alpha + 1)^i} \sum_{j=0}^{i} \sum_{l=0}^{j} \sum_{h=0}^{\infty} \binom{i}{j} \binom{j}{l} \frac{(-1)^l (\theta\alpha)^{j-l} C_{l,h}}{[\Gamma(\lambda)]^l} (\alpha x)^{\beta\lambda + \beta h} e^{-(j-l)(\alpha x)^\beta}, \quad (16)$$

where the coefficients

$$C_{l,h} = (ha_0) \sum\nolimits_{m=1}^{h} [m(h+1) - l] a_m C_{l,h-m}$$

and

$$a_m = \frac{(-1)^l \alpha^{\beta(\lambda+m)}}{m!(\lambda + m)}.$$

After some calculations, finally $E(X_{r:n}^k)$ is given by

$$E(X_{r:n}^k) = \frac{k}{\beta\alpha^k} \sum_{i=n-r+1}^{\infty} \sum_{j=0}^{i} \sum_{l=0}^{j} \sum_{h=0}^{\infty} \binom{i-1}{n-r} \binom{n}{i} \binom{i}{j} \binom{j}{l} (-1)^{i-n+r+l-1}$$

$$\times \frac{(\theta\alpha)^{j-l} C_{l,h}}{(\theta\alpha + 1)^i [\Gamma(\lambda)]^l \cdot (j-l)^{\lambda + h + k/\beta}} \Gamma\left(\lambda + h + \frac{k}{\beta}\right). \quad (17)$$

6. Probability Weighted Moments

The probability weighted moments (PWMs) method can generally be used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly. We calculate the PWMs of the W-GG distribution since they can be used to obtain the moments of the W-GG distribution. The PWMs of a random variable *X* are formally defined by

$$\tau_{s,r} = E[X^s F^r(x)] = \int_0^\infty x^s F^r(x) f(x) dx, \quad (18)$$

where *r* and *s* are positive integers and $F(\cdot)$ and $f(\cdot)$ are the cdf and pdf of the random variable *X*. The PWMs of the W-GG distribution with pdf (6) and cdf (7), are given by

$$\tau_{s,r} = \frac{1}{(\theta\alpha+1)^{r+1}\cdot\alpha^s}\sum_{i=0}^{r}\sum_{j=0}^{r-i}\sum_{h=0}^{\infty}\binom{r}{i}\binom{r-i}{j}C_{i,h}\frac{(-1)^j(\theta\alpha)^{r-i}}{(\Gamma(\lambda))^i(j+1)^{\lambda+h+\frac{s}{\beta}}}$$

$$\times\left\{\frac{\theta\cdot\alpha}{(j+1)}\Gamma\left(\lambda+h+\frac{s}{\beta}+1\right)+\frac{1}{\Gamma(\lambda)\cdot(j+1)^{\lambda}}\Gamma\left(2\lambda+h+\frac{s}{\beta}\right)\right\}, \quad (19)$$

where $C_{i,h}$ and a_i are given in the previous section.

The sth moment of the W-GG distribution can be obtained by putting $r = 0$ in (19).

7. Mean Deviations

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. These are known as the mean deviation about the mean and the mean deviation about the median defined by

$$D_1(X) = \int_0^{\infty}|x-\mu|f(x)dx \text{ and } D_2(X) = \int_0^{\infty}|x-M|f(x)dx,$$

respectively, where $\mu = E(X)$ is the mean of the W-GG distribution and M is the median.

The measures $D_1(X)$ and $D_2(X)$ can be expressed as

$$D_1(X) = 2\mu\cdot F(\mu) - 2T(\mu) \text{ and } D_2(X) = \mu - 2T(M),$$

where $T(z) = \int_0^z xf(x)dx$ is the incomplete mean of X. This integral can be determined from (6) by

$$T(z) = \frac{1}{\theta\alpha+1}\left[\theta\cdot\Gamma\left(\frac{1}{\beta}+1, (\alpha z)^{\beta}\right)+\frac{1}{\alpha\cdot\Gamma(\lambda)}\Gamma\left(\lambda+\frac{1}{\beta}, (\alpha z)^{\beta}\right)\right], \quad (20)$$

where $\Gamma(a, x) = \int_x^{\infty}u^{a-1}e^{-u}du$ is the upper incomplete gamma function.

Thus, the mean deviations $D_1(X)$ and $D_2(X)$ can be easily obtained from (20).

Applications of the mean deviations are the Bonferroni and Lorenz curves. These curves have applications not only in economics to study income and poverty, but also in other fields such as reliability, demography, medicine and insurance. They are defined for given p by

$$B(p) = \frac{T(q)}{p\mu} \text{ and } L(p) = \frac{T(q)}{\mu},$$

respectively, where $q = F^{-1}(p)$ comes directly from the quantile function. In economics, if p is the proportion of units whose income is lower than or equal to q, $L(p)$ gives the proportion of total income volume accumulated by the set of units with an income lower than or equal to q. The Lorenz curve is increasing and convex and given the mean income, the density function of X can be obtained from the curvature of $L(p)$. In a similar manner, the Bonferroni curve $B(p)$ gives the ratio between the mean income of this group and the mean income of the population. In summary, $L(p)$ yields fractions of the total income, while the values of $B(p)$ refer to relative income levels.

8. Rényi and Shannon Entropies

The entropy measure of a random variable X with density function $f(x)$ is a measure of variation of the uncertainty. One of the popular entropy measures is the Rényi entropy given by

$$I_R(\eta) = \left(\frac{1}{1-\eta}\right)\log\left[\int_{\Re} f^{\eta}(x)dx\right],$$

where $\eta > 0, \eta \neq 1.$ For a W-GG distribution with pdf (6), then $f^{\eta}(x)$ is given by

$$f^{\eta}(x) = \left(\frac{\alpha\beta}{\theta\alpha + 1}\right)^{\eta} (\alpha x)^{\eta(\beta - 1)} e^{-\eta(\alpha x)^{\beta}} \left[\theta\alpha + \frac{(\alpha x)^{\beta(\lambda - 1)}}{\Gamma(\lambda)}\right]^{\eta}.$$

Using the expansion

$$(1 + a)^{b} = \sum_{i=0}^{\infty} \binom{b}{i} a^{i}, \quad |\, a \,| < 1,$$

for any b real number and applying it to the term $\left[\theta\alpha + \frac{(\alpha x)^{\beta(\lambda - 1)}}{\Gamma(\lambda)}\right]^{\eta}$, after some simplifications $f^{\eta}(x)$ can be written as:

$$f^{\eta}(x) = \left(\frac{\theta\beta\alpha^{2}}{\theta\alpha + 1}\right)^{\eta} \sum_{i=0}^{\infty} \binom{\eta}{i} \frac{1}{(\theta\alpha)^{i} \cdot [\Gamma(\lambda)]^{i}} (\alpha x)^{\beta(\eta + i\lambda - i) - \eta} \cdot e^{-\eta(\alpha x)^{\beta}}.$$

Then, the Rényi entropy is given by:

$$I_{R}(\eta) = \left(\frac{1}{1 - \eta}\right) \log \left[\left(\frac{\theta\beta\alpha^{2}}{\theta\alpha + 1}\right)^{\eta} \frac{1}{\alpha\beta} \sum_{i=0}^{\infty} \binom{\eta}{i} \frac{\Gamma\left(\eta + i\lambda - i + \frac{1 - \eta}{\beta}\right)}{\eta^{\left(\eta + i\lambda - i + \frac{1 - \eta}{\beta}\right)} [\theta\alpha \cdot \Gamma(\lambda)]^{i}}\right]. \quad (21)$$

The Shannon entropy which is defined by $E[-\log f(x)]$, follows by taking the limit of $I_{R}(\eta)$ as η tends to 1.

9. Residual Life Function

Given that a component survives up to time $y \geq 0$, the residual life is the period beyond y until the time of failure and defined by expectation of the conditional random variable $X \mid X > y$. In reliability, it is well known that the mean residual life function and ratio of two consecutive moments of residual life, determine the distribution uniquely (Gupta and Gupta [16]). Therefore, we obtain the rth order moment of the residual life via the general formula

$$m_r(y) = E[(X-y)^r \mid X > y] = \frac{1}{1-F(x)}\int_y^{\infty}(X-y)^r f(x)dx. \tag{22}$$

Applying the binomial expansion for $(X-y)^r$ and substituting $F(x)$ given by (7) into (22), the rth moment of the residual life of the W-GG distribution is given by

$$m_r(y) = \frac{1}{1-F(y)}\cdot\frac{\alpha^{-i}}{\theta\alpha+1}\sum_{i=0}^{\infty}\binom{r}{i}(-y)^{r-i}$$
$$\times\left\{\theta\alpha\cdot\Gamma\left(\frac{i}{\beta}+1, (\alpha x)^{\beta}\right)+\frac{\Gamma\left(\frac{i}{\beta}+1, (\alpha x)^{\beta}\right)}{\Gamma(\lambda)}\right\}. \tag{23}$$

An expression for the mean residual lifetime function follows by taking $r = 1$.

10. Reversed Residual Life Function

The waiting time since failure is the waiting time elapsed since the failure of an item on condition that this failure had occurred in $[0, y]$. Therefore, we obtain the rth order moment of the reversed residual life via the general formula

$$M_r(y) = E[(y-X)^r \mid X < y] = \frac{1}{F(y)}\int_y^{\infty}(X-y)^r f(x)dx.$$

As doing before, then the rth moment of the reversed residual life of the W-GG distribution is given by

$$M_r(y) = \frac{1}{F(y)}\cdot\frac{\alpha^{-i}}{\theta\alpha+1}\sum_{i=0}^{\infty}\binom{r}{i}(y)^{r-i}(-1)^{i}$$
$$\times\left\{\theta\alpha\cdot\gamma\left(\frac{i}{\beta}+1, (\alpha x)^{\beta}\right)+\frac{\gamma\left(\frac{i}{\beta}+1, (\alpha x)^{\beta}\right)}{\Gamma(\lambda)}\right\}. \tag{24}$$

An expression for the mean reversed residual lifetime function (or, the mean inactivity time) follows by taking $r = 1$.

11. Parameter Estimation

In this section we introduce the method of likelihood estimation as point estimation and the confidence interval as an interval estimation of the unknown parameters then gives the equation used to estimate the parameters using the maximum product spacing estimates and the least square estimates techniques.

11.1. Maximum likelihood method

Let $X_1, X_2, \ldots, X_n$ be a sample of size n from a W-GG distribution. Then the likelihood function (ℓ) is given by:

$$\ell = \left(\frac{\alpha\beta}{\theta\alpha + 1}\right)^n \cdot \prod_{i=1}^{n} (\alpha x_i)^{\beta-1} \cdot e^{-\sum_{i=1}^{n}(\alpha x_i)^\beta} \times \prod_{i=1}^{n}\left[\theta\alpha + \frac{(\alpha x_i)^{\beta(\lambda-1)}}{\Gamma(\lambda)}\right]. \quad (25)$$

Hence, the log-likelihood function $\mathcal{L} = \ln \ell$ becomes:

$$\mathcal{L} = n\ln\alpha + n\ln\beta - n\ln(\theta\alpha + 1) + (\beta - 1)\sum_{i=1}^{n}\ln(\alpha x_i)$$

$$- \sum_{i=1}^{n}(\alpha x_i)^\beta + \sum_{i=1}^{n}\ln\left[\theta\alpha + \frac{(\alpha x_i)^{\beta(\lambda-1)}}{\Gamma(\lambda)}\right]. \quad (26)$$

Therefore, the MLEs of θ, α, β and λ which maximize the function $\mathcal{L}$ must satisfy the following equations:

$$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{-n\alpha}{\theta\alpha + 1} + \alpha\sum_{i=1}^{n}\frac{1}{\theta\alpha + (\alpha x_i)^{\beta(\lambda-1)}/\Gamma(\lambda)}, \quad (27)$$

$$\frac{\partial\mathcal{L}}{\partial\alpha} = \frac{n}{\alpha} - \frac{n\theta}{\theta\alpha + 1} + \frac{n(\beta - 1)}{\alpha} - \beta\alpha^{\beta-1}\sum_{i=1}^{n}x_i^\beta$$

$$+\sum_{i=1}^{n}\frac{\theta\Gamma(\lambda)+(\beta\lambda-\beta)\alpha^{\lambda\beta-\beta-1}x_i^{\beta(\lambda-1)}}{\theta\alpha\Gamma(\lambda)+(\alpha x_i)^{\beta(\lambda-1)}}, \tag{28}$$

$$\frac{\partial\mathcal{L}}{\partial\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\ln(\alpha x_i)-\sum_{i=1}^{n}(\alpha x_i)^{\beta}\log(\alpha x_i)$$

$$+(\lambda-1)\sum_{i=1}^{n}\frac{(\alpha x_i)^{\beta(\lambda-1)}\log(\alpha x_i)}{\theta\alpha+\frac{(\alpha x_i)^{\beta(\lambda-1)}}{\Gamma(\lambda)}} \tag{29}$$

and

$$\frac{\partial\mathcal{L}}{\partial\lambda}=\sum_{i=1}^{n}\frac{(\alpha x_i)^{\beta(\lambda-1)}[\beta\log(\alpha x_i)-\psi(\lambda)]}{\theta\alpha+(\alpha x_i)^{\beta(\lambda-1)}}, \tag{30}$$

where $\psi(z)=\frac{d}{dz}\ln[\Gamma(z)]$ is the Digamma function or Psi function.

The maximum likelihood estimator $\hat{\vartheta}'=(\hat{\theta},\hat{\alpha},\hat{\beta},\hat{\lambda})$ of $\vartheta=(\theta,\alpha,\beta,\lambda)$ is obtained by solving this nonlinear system of (27) through (30). It is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton algorithm to numerically maximize the log-likelihood function. In order to compute the standard error and asymptotic confidence interval we use the usual large sample approximation in which the maximum likelihood estimators can be treated as being approximately trivariate normal.

11.2. Maximum product spacing estimates

The maximum product spacing (MPS) method has been proposed by Cheng and Amin [7]. This method is based on an idea that the differences (spacing) of the consecutive points should be identically distributed. The geometric mean of the differences is given as

$$GM = \sqrt[n+1]{\prod_{i=1}^{n+1} D_i}, \tag{31}$$

where, the difference D_i is defined as

$$D_i = \int_{x_{(i-1)}}^{x_{(i)}} f(x, \theta, \alpha, \beta, \lambda)dx; \quad i = 1, 2, ..., n+1, \tag{32}$$

where $F(x_{(0)}, \theta, \alpha, \beta, \lambda) = 0$ and $F(x_{(n+1)}, \theta, \alpha, \beta, \lambda) = 0$. The MPS estimators $\hat{\theta}_{PS}$, $\hat{\alpha}_{PS}$, $\hat{\beta}_{PS}$ and $\hat{\lambda}_{PS}$ of θ, α, β and λ are obtained by maximizing the geometric mean (GM) of the differences. Substituting pdf of W-GG in (6) and taking logarithm of the above expression, we will have

$$\log GM = \frac{1}{n+1}\sum_{i=1}^{n+1} \log[F(x_{(i)}, \theta, \alpha, \beta, \lambda) - F(x_{(i-1)}, \theta, \alpha, \beta, \lambda)]. \tag{33}$$

The MPS estimators $\hat{\theta}_{PS}$, $\hat{\alpha}_{PS}$, $\hat{\beta}_{PS}$ and $\hat{\lambda}_{PS}$ of θ, α, β and λ can be obtained as the simultaneous solution of the following non-linear equations:

$$\frac{\partial \log GM}{\partial \theta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_{\theta}(x_{(i)}, \theta, \alpha, \beta, \lambda) - F'_{\theta}(x_{(i-1)}, \theta, \alpha, \beta, \lambda)}{F(x_{(i)}, \theta, \alpha, \beta, \lambda) - F(x_{(i-1)}, \theta, \alpha, \beta, \lambda)}\right] = 0,$$

$$\frac{\partial \log GM}{\partial \alpha} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_{\alpha}(x_{(i)}, \theta, \alpha, \beta, \lambda) - F'_{\alpha}(x_{(i-1)}, \theta, \alpha, \beta, \lambda)}{F(x_{(i)}, \theta, \alpha, \beta, \lambda) - F(x_{(i-1)}, \theta, \alpha, \beta, \lambda)}\right] = 0,$$

$$\frac{\partial \log GM}{\partial \beta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_{\beta}(x_{(i)}, \theta, \alpha, \beta, \lambda) - F'_{\beta}(x_{(i-1)}, \theta, \alpha, \beta, \lambda)}{F(x_{(i)}, \theta, \alpha, \beta, \lambda) - F(x_{(i-1)}, \theta, \alpha, \beta, \lambda)}\right] = 0,$$

and

$$\frac{\partial \log GM}{\partial \lambda} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_{\lambda}(x_{(i)}, \theta, \alpha, \beta, \lambda) - F'_{\lambda}(x_{(i-1)}, \theta, \alpha, \beta, \lambda)}{F(x_{(i)}, \theta, \alpha, \beta, \lambda) - F(x_{(i-1)}, \theta, \alpha, \beta, \lambda)}\right] = 0,$$

where

$$F'_{\theta}(x_{(i)}, \alpha, \beta, \lambda, \theta) = \frac{\alpha}{(\theta\alpha+1)^2}(1 - e^{-(\alpha x)^{\beta}})$$

$$+ \frac{\alpha}{(\theta\alpha+1)\Gamma(\lambda)}\sum_{i=0}^{\infty}(-1)^{i+1}\frac{(\alpha x)^{\beta(\lambda+i)}}{i!(\lambda+i)},$$

$$F'_{\alpha}(x_{(i)}, \alpha, \beta, \lambda, \theta) = \frac{\theta}{(\theta\alpha+1)^2}(1 - e^{-(\alpha x)^{\beta}}) + \frac{\theta\beta}{\theta\alpha+1}(\alpha x)^{\beta}e^{-(\alpha x)^{\beta}}$$

$$+ \frac{\theta}{(\theta\alpha+1)^2 \cdot \Gamma(\lambda)}\sum_{i=0}^{\infty}(-1)^{i+1}\frac{(\alpha x)^{\beta(\lambda+i)}}{i!(\lambda+i)}$$

$$+ \frac{\beta}{\alpha(\theta\alpha+1)\Gamma(\lambda)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}(\alpha x)^{\beta(\lambda+i)},$$

$$F'_{\beta}(x_{(i)}, \alpha, \beta, \lambda, \theta) = \frac{1}{\theta\alpha+1}(\alpha x)^{\beta}\ln(\alpha x)$$

$$\cdot\left\{\theta\alpha e^{-(\alpha x)^{\beta}} + \frac{1}{\Gamma(\lambda)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}(\alpha x)^{\beta(\lambda+i-1)}\right\},$$

and

$$F'_{\lambda}(x_{(i)}, \alpha, \beta, \lambda, \theta) = \frac{\psi(\lambda)}{(\theta\alpha+1)\Gamma(\lambda)} \cdot \sum_{i=0}^{\infty}(-1)^{i+1}\frac{(\alpha x)^{\beta(\lambda+i)}}{i!(\lambda+i)}$$

$$+ \frac{1}{(\theta\alpha+1)\Gamma(\lambda)}\sum_{i=0}^{\infty}\frac{(\alpha x)^{\beta(\lambda+i)}[\beta(\lambda+i)\ln(\alpha x) - i]}{i!(\lambda+i)^2}.$$

11.3. Least square estimates

Let $x_{(1)}, x_{(2)}, ..., x_{(n)}$ be the ordered sample of size n drawn from the W-GG distribution population pdf. Then, the expectation of the empirical cumulative distribution function is defined as

$$E[F(X_{(i)})] = \frac{i}{n+1}; \quad i = 1, 2, ..., n. \tag{34}$$

The least square estimates (LSEs) $\hat{\alpha}_{LS}$, $\hat{\beta}_{LS}$, $\hat{\lambda}_{LS}$ and $\hat{\theta}_{LS}$ of α, β, λ and θ are obtained by minimizing

$$Z(\alpha, \beta, \lambda, \theta) = \sum_{i=1}^{n}\left[F(x_{(i)}, \alpha, \beta, \lambda, \theta) - \frac{i}{n+1}\right]^2.$$

Therefore, $\hat{\alpha}_{LS}$, $\hat{\beta}_{LS}$, $\hat{\gamma}_{LS}$ and $\hat{\theta}_{LS}$ of α, β, γ and θ can be obtained as the solution of the following system of equations:

$$\frac{\partial Z(\alpha, \beta, \lambda, \theta)}{\partial \alpha} = \sum_{i=1}^{n} F'_{\alpha}(x_{(i)}, \alpha, \beta, \lambda, \theta)\left(F(x_{(i)}, \alpha, \beta, \lambda, \theta) - \frac{i}{n+1}\right) = 0,$$

$$\frac{\partial Z(\alpha, \beta, \lambda, \theta)}{\partial \beta} = \sum_{i=1}^{n} F'_{\beta}(x_{(i)}, \alpha, \beta, \lambda, \theta)\left(F(x_{(i)}, \alpha, \beta, \lambda, \theta) - \frac{i}{n+1}\right) = 0,$$

$$\frac{\partial Z(\alpha, \beta, \lambda, \theta)}{\partial \lambda} = \sum_{i=1}^{n} F'_{\lambda}(x_{(i)}, \alpha, \beta, \lambda, \theta)\left(F(x_{(i)}, \alpha, \beta, \lambda, \theta) - \frac{i}{n+1}\right) = 0,$$

and

$$\frac{\partial Z(\alpha, \beta, \lambda, \theta)}{\partial \theta} = \sum_{i=1}^{n} F'_{\theta}(x_{(i)}, \alpha, \beta, \lambda, \theta)\left(F(x_{(i)}, \alpha, \beta, \lambda, \theta) - \frac{i}{n+1}\right) = 0.$$

These non-linear can be routinely solved using Newton's method or fixed point iteration techniques.

12. Application

In this section, we use simulated and real data sets to compare the fits of the new model and illustrate the usefulness of the new model.

12.1. Simulation study

To assess the behavior of the maximum likelihood estimators of the parameters θ, α, β and λ under the finite samples, we construct a Monte Carlo simulation. All results were obtained from 2000 Monte Carlo replications and the simulations were carried out using the statistical software package R. In each replication a random sample of size n is drawn from the new distribution, the L-BFGS-B method has been used to maximize the log-likelihood function, and the inversion method has been used to generate random samples from the distribution. The true parameter values used in the data generating processes are $\theta = 0.6, \theta = 0.8, \beta = 3$ and $\lambda = 8.$ Table 2 presents the mean maximum likelihood estimates of the parameters and mean squared errors (MSE) for sample sizes $n = 50,$ $n = 80$ and $n = 100.$

Table 2. Mean estimates and mean squared errors of $\hat{\theta}, \hat{\alpha}, \hat{\beta}$ and $\hat{\lambda},$ the maximum likelihood estimators of the parameters

n	Parameters	Mean	Var	Bias	MSE
50	$\hat{\theta}$	0.5991068	0.0476096	0.000893	0.047610
	$\hat{\alpha}$	0.8004673	0.0118408	–0.000467	0.011841
	$\hat{\beta}$	3.370479	31.8344	–0.370479	31.97165
	$\hat{\lambda}$	8.257732	3.996582	–0.257732	4.063008
50	$\hat{\theta}$	0.61623	0.00344276	–0.01623	0.034691
	$\hat{\alpha}$	0.804817	0.0057155	–0.004817	0.005739
	$\hat{\beta}$	3.071927	0.3319827	–0.071927	0.337156
	$\hat{\lambda}$	8.268677	1.925715	0.072187	1.997902
100	$\hat{\theta}$	0.6102073	0.0274986	–0.010207	0.027603
	$\hat{\alpha}$	0.8051835	0.0045103	–0.005184	0.004537
	$\hat{\beta}$	3.055998	0.2694269	–0.055998	0.272563
	$\hat{\lambda}$	8.226823	1.593101	–0.226823	1.6445497

Based on the results in the previous table, we notice that the biases and root mean squared errors of the maximum likelihood estimators of θ, α, β and λ decay toward zero as the sample size increases.

12.2. Aarset data – uncensored

The data set is given by Aarset [1] represents lifetimes of 50 industrial devices put on life test at time zero. These data reported in Mudholkar and Srivastava [18], Mudholkar et al. [19] and Wang [27] exhibit a bathtub-shaped failure rate property. The data is given by: 0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86. We compare the fitting of the ETWG model with 5 nested models and 8 non-nested models. In each case, the parameters are estimated by maximum likelihood as described in Section 11, using the SAS code (PROC NLMIXED).

We have fitted the mixture Weibull generalized gamma (W-GG) distribution to the data set using MLE, and compared this model with generalized gamma (GG) (Stacy [26]), Weibull (W) (Weibull [28]), Lindley (L), two parameter Lindley (2-PL) (Shanker et al. [23]) and Quasi Lindley (QL) (Shanker and Mishra [22]) distributions.

The model selection is carried out using the Akaike information criterion (AIC), the Bayesian information criterion (BIC) and the consistent Akaike information criteria (AICC) using

$$\mathrm{AIC} = -2\mathcal{L} + 2k,$$

$$\mathrm{AIC_C} = -2\mathcal{L} + \left(\frac{2kn}{n-k-1}\right),$$

and

$$\mathrm{BIC} = -2\mathcal{L} + k\log(n),$$

where $\mathcal{L}$ denotes the log-likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters, and n is the sample size.

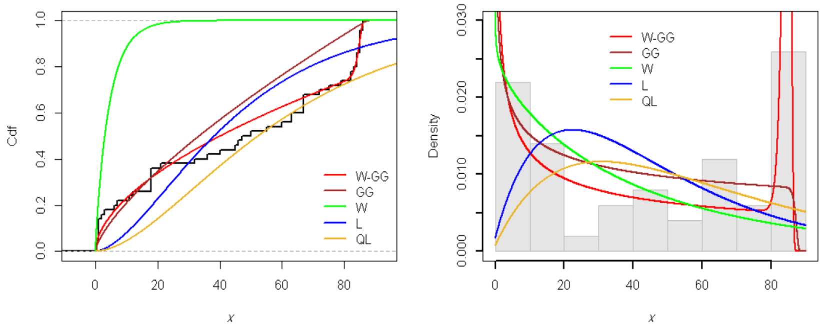
Table 2 lists the MLEs of the parameters (the standard errors are given in parentheses) for the GG, W, L, 2P-L and QL distributions fitted to the data and the values of the AIC or AIC_C or BIC statistic. These numerical results are obtained using the SAS (PROC NLMIXED). Based on the criterion, we conclude that the W-GG distribution provides a superior fit to these data than the other models.

Table 3. MLEs (standard errors in parentheses) and the measures AIC, AIC_C and BIC to Aarset data for nested models.

Model	Parameter				-2LL	AIC	AIC_C	BIC
	α	β	λ	θ				
W-GG	0.01177 (0.000045)	88.1573 (24.8174)	0.006344 (0.001999)	27.0067 (9.5700)	407.9	415.9	416.8	423.6
GG	0.01143 (0.000213)	148.55 (133.05)	0.004849 (0.004365)		441.2	447.2	447.7	452.9
W	0.02227 (0.003443)	0.9490 (0.1196)			482.0	486.0	486.3	489.8
L	0.04288 (0.004290)				502.9	504.9	504.9	506.8
2P-L	0.03167 (0.005316)			$\eta = 0.02559$ (0.0224)	480.3	484.3	484.6	488.2
QL	0.03167 (0.005316)			$\eta = 1.2377$ (0.9367)	480.3	484.3	484.6	488.2

The previous table shows that W-GG distribution fitted the data better than the other models.

In order to assess if the model is appropriate, we plot in Figures 3 (a) and (b) the histogram of the data and the W-GG, GG, W, L and QL distributions and the empirical and their estimated cdf functions, respectively. These plots indicate that the W-GG distribution provides a better fit to these data than all its sub-models.



(a) (b)

Figure 3. (a) Estimated densities of the W-GG, GG, W, L, and QL distributions for the data. (b) Estimated cdf function from the fitted W-GG, GG, W, L and QL distributions and the empirical cdf for the data.

In addition, Figures 4 (a), (b), (c), (d), (e) and (f) present the probability-probability (P-P) plot for the W-GG, GG, W, L, QL and 2P-L distributions which specified used to determine how well a specific distribution fits to the observed data. This plot will be approximately linear if the specified theoretical distribution is the correct model.

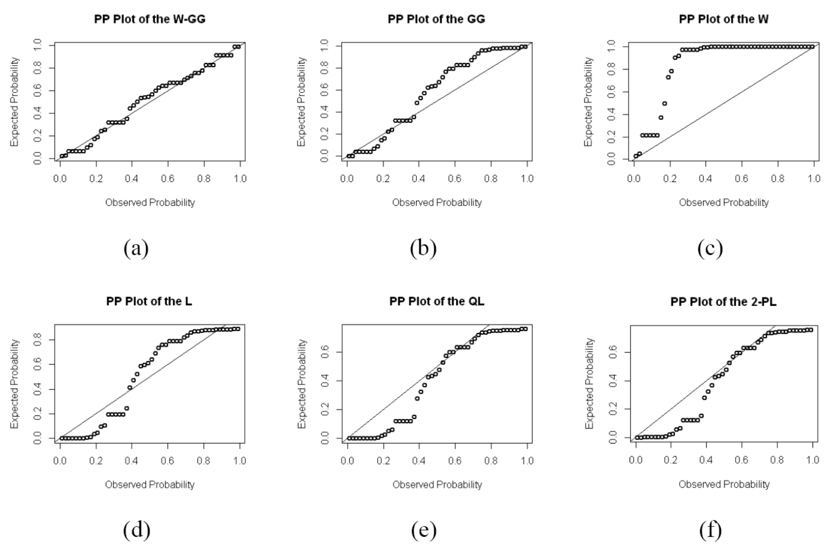


Figure 4. (a), (b), (c), (d), (e) and (f) are the P-P plot for the W-GG, GG, W, L, 2-PL and QL distributions respectively.

In the following, we shall compare the proposed model with several other lifetime distributions based on the Aarset data.

• **The reduced new modified Weibull (RNMW) distribution introduced by Almalki [2].** The pdf of RNMW distribution (with three parameters α, β and λ) is

$$f(x) = \frac{1}{2\sqrt{x}}[\alpha + \beta(1 + 2\lambda x)e^{\lambda x}]e^{-\alpha\sqrt{x} - \beta\sqrt{x}e^{\lambda x}}, \quad x > 0,$$

where $\alpha, \beta > 0$ are scale parameters and $\lambda > 0$ is an acceleration parameter.

• **The new modified Weibull (NMW) distribution introduced by Almalki and Yuan [3].** The pdf of NMW distribution (with five parameters $\alpha, \theta, \beta, \gamma$ and λ) is

$$f(x) = [\alpha\theta x^{\theta-1} + \beta(\gamma + \lambda x)x^{\gamma-1}e^{\lambda x}]e^{-\alpha x^{\theta} - \beta x^{\gamma}e^{\lambda x}}, \ x > 0,$$

where $\theta, \gamma > 0$ are shape parameters, $\alpha, \beta > 0$ are scale parameters and $\lambda > 0$ is an acceleration parameter.

• **The additive Weibull (AW) distribution introduced by Xie and Lie [29].** The pdf of NMW distribution (with five parameters α, θ, β and γ) is

$$f(x) = [\alpha\theta x^{\theta-1} + \beta\gamma x^{\gamma-1}]e^{-\alpha x^{\theta} - \beta x^{\gamma}}, \quad x > 0,$$

where $\theta, \gamma > 0$ are shape parameters and $\alpha, \beta > 0$ are scale parameters.

• **The Kumaraswamy generalized gamma (Kw-GG) distribution introduced by de Pascoa [13].** The pdf of Kw-GG distribution (with five parameters α, τ, k, λ and φ) is

$$f(x) = \frac{\lambda\varphi\tau}{\alpha\Gamma(k)}\left(\frac{t}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{t}{\alpha}\right)^{\tau k}} \left[P\left(k, \left(\frac{t}{\alpha}\right)^{\tau}\right)\right]^{\lambda-1}$$

$$\cdot \left\{1 - \left[P\left(k, \left(\frac{t}{\alpha}\right)^{\tau}\right)\right]^{\lambda}\right\}^{\varphi-1}, \ x > 0,$$

where $\alpha > 0$ is a scale parameter and $\tau, k, \lambda, \varphi > 0$ are shape parameters.

• **The beta generalized Weibull (B-GW) distribution introduced by Singla et al. [25].** The pdf of BGW distribution (with five parameters α, τ, k, λ and φ) is

$$f(x) = \frac{\alpha\beta\lambda^{\beta}t^{\beta-1}}{B(a, b)} e^{-(\lambda t)^{\beta}} (1 - e^{-(\lambda t)^{\beta}})^{\alpha a-1} [1 - (1 - e^{-(\lambda t)^{\beta}})^{\alpha}]^{b-1}, \ x > 0,$$

where $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}$ is the beta function, $\lambda > 0$ is a scale parameter and $a, b, \alpha, \beta > 0$ are shape parameters.

• **The beta Gompertz (B-G) distribution introduced by Jafari et al. [15].** The pdf of BG distribution (with five parameters α, τ, k, λ and φ) is

$$f(x) = \frac{\theta e^{\gamma x}}{B(a, b)} e^{\frac{\theta b}{\gamma}(e^{\gamma x}-1)} \left[1 - e^{\frac{\theta}{\gamma}(e^{\gamma x}-1)}\right]^{a-1}, \ x > 0,$$

where $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}$ is the beta function, $\gamma, \theta, b > 0$ are scale parameters and $a > 0$ is a shape parameter.

• **The Kumaraswamy Weibull (Kw-W) distribution introduced by Cordeiro et al. [10].** The pdf of BG distribution (with five parameters α, τ, k, λ and φ) is

$$f(x) = ab\alpha\gamma x^{\gamma-1} e^{-\alpha x^{\gamma}} [1 - e^{-\alpha x^{\gamma}}]^{a-1} \{1 - [1 - e^{-\alpha x^{\gamma}}]^{a}\}^{b-1}, \ x > 0,$$

where $\alpha > 0$ is a scale parameter and $a, b, \gamma > 0$ are shape parameters.

• **The exponentiated generalized gamma (EGG) distribution introduced by Cordeiro et al. [10]-[11].** The pdf of BG distribution (with four parameters α, τ, k and λ) is

$$f(x) = \frac{\lambda\tau}{\alpha\Gamma(k)} \left(\frac{t}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{t}{\alpha}\right)^{\tau k}} \left[P\left(k, \left(\frac{t}{\alpha}\right)^{\tau}\right)\right]^{\lambda-1}, \ x > 0,$$

where $\alpha > 0$ is a scale parameters and $a, b, \gamma > 0$ are shape parameters.

• **The beta XTG (B-XTG) distribution introduced by Cordeiro et al. [12].** The pdf of BW distribution (with four parameters a, b, α, β and λ) is

$$f(x) = \frac{\alpha\beta}{B(a, b)}\left(\frac{t}{\lambda}\right)^{\beta} e^{\left(\frac{t}{\lambda}\right)^{\beta}}\left[1 - e^{-\lambda e^{\left[\left(\frac{t}{\lambda}\right)^{\beta} - 1\right]}}\right]^{a-1} e^{-\alpha b\lambda e^{\left[\left(\frac{t}{\lambda}\right)^{\beta} - 1\right]}}, \; x > 0,$$

where $\lambda > 0$ is a scale parameter and $a, b, \beta > 0$ are shape parameters.

Table 4. MLEs (standard errors in parentheses) and the measures AIC, AIC_C and BIC to Aarset data for non-nested models

Model	Parameter					-2LL	AIC	AIC_C	BIC
W-GG	0.01177 (0.000045)	88.1573 (24.8174)	0.006344 (0.001999)	27.0067 (9.5700)		407.9	415.9	416.8	423.6
RNMW	$\alpha = 0.102$ (0.031)	$\beta = 7.015 \times 10^{-8}$ 1.501×10^{-7}	$\lambda = 0.180$ (0.020)			427.1	433.1	433.6	438.8
NMW	$\alpha = 0.071$ (0.031)	$\beta = 7.015 \times 10^{-8}$ 1.501×10^{-7}	$\gamma = 0.016$ (3.602)	$\theta = 0.595$ (0.128)	$\lambda = 0.197$ (0.184)	425.8	435.8	437.2	445.4
Kw-GG	$\alpha = 84.5056$ (0.2099)	$\tau = 79.5358$ (2.0929)	$k = 0.0080$ (0.0021)	$\lambda = 0.5393$ (0.2387)	$\varphi = 0.3431$ (0.0565)	413.1	423.1	424.5	432.7
EGG	$\alpha = 86.5056$ (0.5539)	$\tau = 28.8487$ (0.0308)	$k = 1.0314$ (0.0001)	$\lambda = 0.0272$ (0.0039)		413.1	455.7	456.6	463.4
B-GW	$\alpha = 0.587$ ()	$b = 0.315$ ()	$\lambda = 0.016$ ()	$\alpha = 0.136$ ()	$\beta = 5.64$ ()	440.7	450.7	452.1	460.3
B-G	$\alpha = 0.2158$ (0.0392)	$\beta = 0.2467$ (0.0448)	$\theta = 0.0003$ (0.0001)	$\gamma = 0.0882$ (0.0030)		441.3	449.3	450.2	456.9
Kw-W	$\alpha = $ 1e-8 (0.000)	$\gamma = 4.4716$ (0.0941)	$a = 0.0706$ (0.0021)	$\lambda = 0.5393$ (0.0257)	$b = 0.2370$ (0.0762)	443.0	453.0	454.4	462.6
B-XTG	$a = 0.1031$ (0.0213)	$b = 0.1196$ (0.0535)	1E-8 (0.0000)	$\beta = 0.3036$ (0.0179)	$\lambda = 0.0016$ (0.0012)	441.1	451.1	452.5	460.7

12.3. Efron - censored

The following data represents the survival times in days of head and neck cancer patients after a treatment considered earlier by Efron (1988), these data reported in Cooray [9]. This clinical trial data consist of $n = 51$ patients with radiation therapy alone denoted by arm A. Nine patients were lost to follow-up and were regarded as right-censored.

Arm A: 7, 34, 42, 63, 64, 74+, 83, 84, 91, 108, 112, 129, 133, 133, 139, 140, 140, 146, 149, 154, 157, 160, 160, 165, 173, 176, 185+, 218, 225, 241, 248, 273, 277, 279+, 297, 319+, 405, 417, 420, 440, 523, 523+, 583, 594, 1101, 1116+, 1146, 1226+, 1349+, 1412+, 1417, where (+) indicates right-censored observation.

Table 5. MLEs and the measures AIC and $\mathrm{AIC_X}$ to Arm data.

Models	Estimates				Measures		
	θ	α	β	λ	-2LL	AIC	$\mathrm{AIC_C}$
W-G	**0.0000**	**67.7443**	**0.25693**	**12.7014**	**581.9812**	**589.9812**	**590.85**
Weibull		427.3736	0.93595		589.3286	593.3286	593.58
R		0.001725			651.8206	653.8206	653.90
Exponential		0.002428			589.7794	591.7794	591.56
2P-L	1.83442e-03	1.72494e-03			651.8206	655.8206	656.07
L		0.0049219			605.3936	607.3936	607.48

The next figure shows the estimated distribution function of the empirical cdf with a lower and upper confidence interval for the values which shows a great fitting to the data.

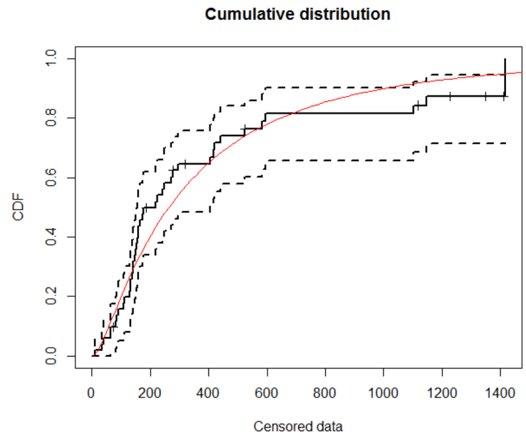


Figure 5. The estimated cdf function from the fitted of new distribution and the empirical cdf for the data with confidence interval of it.

Concluding Remarks

There has been a great interest among statisticians and applied researchers in constructing flexible lifetime models to facilitate better modelling of survival data. Consequently, a significant progress has been made towards the generalization of some well-known lifetime models and their successful applications to problems in several areas. In this paper, we introduce a new four-parameter distribution obtained using mixture distribution. We refer to the new model as the W-GG distribution and study some of its mathematical and statistical properties. We provide the pdf, the cdf and the hazard rate function of the new model, explicit expressions for the moments, and mean deviations. The residual lifetime, reversed residual lifetime, moment of order statistics, probability weighted moments and the measures of entropies Rényi and Shannon entropies functions are introduced. The model parameters are estimated by maximum likelihood. The new model is compared with nested and non nested models and provides consistently better fit than other classical lifetime models. We hope that the proposed distribution will serve as an alternative model to other models available in the literature for modelling positive real data in many areas such as engineering, survival analysis, hydrology and economics.

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